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A MINORANT–MAJORANT APPROXIMATION METHOD FOR THE SOLUTION OF THE NONLINEAR FOURIER EQUATION

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Abstract—In the course of an investigation of the nonlinear Fourier equation of heat conduction by a method of analytic continuation with respect to the temperature maximum y_m as the continuation parameter, it is shown how positive majorants and positive minorants (which are defined to vary with the parameter y_m) can be used to furnish continuous upper and lower bounds of a function $\lambda(y_m)$ which governs the intensity of the generation of heat in the course of a certain exothermic process. The function $\lambda(y_m)$ constitutes a characteristic property of the problem because it contains information as to the existence of stable, unstable and critical temperature distributions which result from the boundary-value problem of Fourier's equation. In order to provide a test for the proposed method, continuous upper and lower bounds of $\lambda(y_m)$ are derived for a Fourier problem of which the exact numerical solution is known.

NOMENCLATURE

Ω,	region in which the exothermic process takes place;
Γ,	boundary of Ω;
$\hat{L}$,	uniformly elliptic differential operator describing the heat transport within Ω;
$\hat{B}$,	linear homogeneous operator describing the heat transfer across the boundary Γ of Ω;
$\mathbf{x}$,	position vector of point in $\Omega+\Gamma$;
$y(\mathbf{x})$,	temperature distribution in $\Omega+\Gamma$;
y_m,	maximum value of the temperature;
$\mathbf{x}_0$,	symmetry-determined location of the temperature maximum y_m;
$W(y)$,	function representing the temperature dependence of the generation of heat in $\Omega\,(W_y = \partial W/\partial y, y' = \partial y/\partial\delta, \dot{y} = \partial y/\partial\varepsilon)$;
λ,	parameter governing the intensity of the heat generation given by $W(y)$;
μ, φ,	eigenvalue and eigenfunction;
ε, δ,	real continuation parameters;
$\hat{G}$,	integral operator;
$G(\mathbf{x}, \mathbf{x}\dagger)$,	kernel of $\hat{G}$ (Green's function);
$m(y, y_m)$,	minorant of $W(y)$ on $0 \leqq y \leqq y_m$;
$M(y, y_m)$,	majorant of $W(y)$ on $0 \leqq y \leqq y_m$;
$t(y, y_m)$,	tangent of $W(y)$ at y_m;
$s(y, y_m)$,	secant of $W(y)$ at y_m;
λ_m,	parameter governing the intensity of the heat generation given by $m(y, y_m)$;
λ_M,	parameter governing the intensity of the heat generation given by $M(y, y_m)$.

1. INTRODUCTION

THE STATIONARY thermal states of an open thermodynamic system evolving in an exothermic irreversible process are of considerable interest. The generation of heat in such a system may be caused by a homogeneous exothermic chemical reaction [1] or may be due to electric [2] or viscous [3] resistance. A stationary thermal state of the system will occur if the heat produced within the system on account of the exothermic nature of the process is balanced by the heat transferred into the surroundings of the system.

Let Ω designate the region occupied by the thermodynamic system, with Γ designating the boundary of Ω. Then the stationary thermal state of the system is characterized by a temperature distribution y which results as the solution of the following, usually strongly nonlinear boundary-value problem of Fourier's equation:

$$\hat{L}(y) + \lambda W(y) = 0 \quad \text{on } \Omega,$$
$$\hat{B}(y) = 0 \quad \text{on } \Gamma. \tag{1}$$

Here, $\hat{L}$ designates the uniformly elliptic differential operator connected with the heat transport within Ω, $\hat{B}$ designates a linear homogeneous operator describing the heat transfer across the boundary Γ of Ω. The generation of heat in Ω which is due to the exothermic nature of the thermodynamic process under consideration, is designated by the expression $\lambda W(y)$ depending on the temperature y and on a real parameter $\lambda (\lambda \geqq 0)$, which is to represent the experimenter's control of the intensity of the generation of heat. From these definitions derives the following property of $W(y)$:

$$W(y) > 0 \quad \text{for} \quad y \geqq 0. \tag{2}$$

For the operators $\hat{L}$ and $\hat{B}$, the usual assumptions are made [4–6], such that the maximum principle holds [7], whereby the solutions y of equation (1) are positive on Ω, and such that for a solution (y_0, λ_0) of

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(1), the derived operator

$$(\hat{L} + \lambda_0 W_y(y_0), \hat{B}) \tag{3}$$

constitutes a Fredholm operator [6].

With these assumptions made, the solution of problem (1) may be derived with various methods (see [5] for a list of references). On account of the nonlinear dependence of W on y, one finds it difficult (excepting a few rare cases, comp. [17]) to derive exact solutions (y, λ) of (1). But before embarking on some difficult mathematical deliberations, it seems to be advisable to recall a physical fact which is of importance in connection with problem (1): because of an incomplete theory of the physical phenomenon under investigation, the boundary conditions embodied in $\hat{B}(y) = 0$ on Γ represent a drastic approximation (e.g. Newton's law of cooling) and usually the same holds for the function $W(y)$ representing the generation of heat. $W(y)$ may have even been found by interpolation from empirical data. It is therefore to be expected that the given function $W(y)$ represents the generation of heat approximately and only on a finite temperature interval $0 \leqq y \leqq \bar{y}$. Thus it seems to be well worth trying to find a simple approximation method by which the salient features of the solutions of problem (1) (e.g. existence, stability, criticality, etc.) are preserved and by which the solutions of problem (1) are restricted from the start to the interval where $W(y)$ is known. In the following text, such an approximation method is proposed which has already been applied to a few examples by this author [8, 9]. The method applies to problem (1) irrespective of whether $\hat{L}$ is in the form of an ordinary or a partial differential operator and it consists in finding the solutions of appropriately defined soluble boundary value problems related to problem (1).

2. THE TEMPERATURE MAXIMUM DEFINED AS THE CONTINUATION PARAMETER

Quite often, the experimenter is able to select the shape of the boundary Γ and to impose a heat transfer across Γ such that the symmetry of the boundary-value problem (1) fixes the position of the maximum y_m of the temperature distribution y on $\Omega + \Gamma$ (e.g. comp. [1] and [10], where a list of symmetric boundary shapes may be found). It is then possible to consider y_m as an independent parameter whereby the solutions of problem (1) result in the following parametric representation (e.g. comp. [1]):

$$(y(\mathbf{x}, y_m), \lambda(y_m)). \tag{4}$$

Here, $\mathbf{x}$ designates a position in $\Omega + \Gamma$. If $\mathbf{x}_0$ represents the symmetry determined position of the maximum y_m of the temperature distribution y, then one must have:

$$y(\mathbf{x}_0, y_m) = y_m. \tag{5}$$

Let (y_0, λ_0) be a solution of problem (1) and let μ_0 be the principal eigenvalue [5] of the boundary-value problem of the linear derived operator given by equation (3):

$$\hat{L}(\varphi) + \lambda_0 W_y(y_0)\varphi + \mu\varphi = 0 \quad \text{on } \Omega,$$
$$\hat{B}(\varphi) = 0 \quad \text{on } \Gamma. \tag{6}$$

Then there exists a single eigenfunction φ_0 related to the principal eigenvalue μ_0 and φ_0 does not change its sign on Ω. If $\mu_0 > 0$, then (y_0, λ_0) is a stable solution, if $\mu_0 < 0$, then (y_0, λ_0) is an unstable solution and if $\mu_0 = 0$, then (y_0, λ_0) is a critical solution (a critical point) of problem (1) (comp. [5]). A critical point of problem (1) represents a particular branching point of (1) in that a critical point separates a branch of stable solutions from a branch of unstable solutions, whereas an ordinary branching point separates branches of unstable solutions of (1). It is perhaps worthwhile stating that unstable stationary temperature distributions may never be obtained in an experiment. A stable or a critical solution of problem (1) may be analytically continued (uniquely) with respect to the parameter y_m by the perturbation method proposed in [11], by defining ε as the perturbation parameter:

$$\varepsilon = y_m - y_{m0} \tag{7}$$

with y_{m0} designating the maximum of y_0. The function $\lambda(y_m)$ may be seen as a 'response function' of the system [8, 9]. A critical solution of problem (1) is connected with an extremum of $\lambda(y_m)$ but the inverse of this proposition does not necessarily hold (comp. [12] and the discussion of Steggerda's results by [1]). The application of the perturbation method proposed by [11] furnishes the continuation of a given solution (y_0, λ_0) with respect to ε given by equation (7) in the following form:

$$y(\mathbf{x}, \varepsilon) = y_0(\mathbf{x}) + \dot{y}(\mathbf{x})\varepsilon + \tfrac{1}{2}\ddot{y}(\mathbf{x})\varepsilon^2 + \cdots$$

$$\lambda(\varepsilon) = \lambda_0 + \dot{\lambda}\varepsilon + \tfrac{1}{2}\ddot{\lambda}\varepsilon^2 + \cdots \tag{8}$$

It is easily derived that for a stable solution (y_0, λ_0) holds that:

$$\dot{y} > 0, \qquad \dot{\lambda} > 0 \tag{9}$$

and that for a critical solution (y_0, λ_0) holds that

$$\dot{y} > 0, \qquad \dot{\lambda} = 0. \tag{10}$$

This perturbation theory has been worked out completely elsewhere [13].

Because of the definition of the linear operators $\hat{L}$ and $\hat{B}$, one obtains for $\lambda_0 = 0$ the unique stable solution $y_0 \equiv 0$ of problem (1) and $(y_0 \equiv 0, \lambda_0 = 0)$ may be continued with respect to y_m. Therefore, the function $\lambda(y_m)$ consists of at least one branch emerging from $\lambda_0 = 0$, $y_{m0} = 0$ such that this branch corresponds to a branch of stable solutions of (1). If a critical point exists for this branch, then it corresponds to a maximum of $\lambda(y_m)$,

3. POSITIVE MINORANTS AND MAJORANTS OF $W(y)$

For the introduction of minorants and majorants of

$W(y)$, the following nonlinear boundary-value problem turns out to be of interest:

$$\hat{L}(y) + \lambda(f(y) + \delta g(y)) = 0 \quad \text{on } \Omega,$$
$$\hat{B}(y) = 0 \quad \text{on } \Gamma. \tag{11}$$

Here, λ and δ are real parameters, $\hat{L}$ and $\hat{B}$ are the same linear operators as in problem (1), $f(y)$ and $g(y)$ are two real valued functions which—for the sake of the perturbation method–are supposed to be analytic in y. For λ and δ, it is supposed to hold that:

$$\lambda \geqq 0 \qquad 0 \leqq \delta \leqq 1, \tag{12}$$

$$f(y) + \delta g(y) > 0 \quad \text{for } 0 \leqq \delta \leqq 1, \quad y \geqq 0. \tag{13}$$

Let $(y_0, \lambda_0, \delta_0)$ be a solution of problem (11). On account of the maximum principle [7], one finds that:

$$y_0 > 0 \quad \text{on } \Omega, \qquad y_0 \geqq 0 \quad \text{on } \Omega + \Gamma \tag{14}$$

and that y_0 does not have a minimum on Ω so that y_0 possesses a unique maximum y_{m0} on $\Omega + \Gamma$. In the following derivation, δ is considered an independent parameter of problem (11) and the solution (y, λ_0, δ_0) is analytically continued with respect to $(\delta - \delta_0)$. Designating the differentiation with respect to δ by dashes, e.g.:

$$y' \equiv \left.\frac{\partial y}{\partial \delta}\right|_{\delta=\delta_0}, \qquad \lambda' \equiv \left.\frac{\partial \lambda}{\partial \delta}\right|_{\delta=\delta_0} \quad \text{etc.} \tag{15}$$

one obtains the following recursive system of boundary-value problems by inserting in (11) the series expansions in $(\delta - \delta_0)$ of y and λ:

zeroth order

$$\hat{L}(y_0) + \lambda_0(f(y_0) + \delta_0 g(y_0)) = 0 \quad \text{on } \Omega,$$
$$\hat{B}(y_0) = 0 \quad \text{on } \Gamma \tag{16a}$$

first order

$$\hat{L}(y') + \lambda_0(f_y(y_0) + \delta_0 g_y(y_0))y' + \lambda_0 g(y_0)$$
$$+ \lambda'(f(y_0) + \delta_0 g(y_0)) = 0 \quad \text{on } \Omega,$$
$$\hat{B}(y') = 0 \quad \text{on } \Gamma. \tag{16b}$$

The higher orders of the perturbation system may be obtained in the usual fashion. Because of the symmetry of problem (11), the maximum y_m of the solution y is located at the position $\mathbf{x}_0$. The following constraints are imposed on (y, λ, δ):

$$y'(\mathbf{x}_0) = 0, \quad y''(\mathbf{x}_0) = 0, \ \ldots \tag{17}$$

whereby:

$$y(\mathbf{x}_0, \delta - \delta_0) \equiv y_0(\mathbf{x}_0) = y(\mathbf{x}_0, 0) = y_{m0}. \tag{18}$$

If $(y_0, \lambda_0, \delta_0)$ represents a stable solution of (11), then there exists a bounded integral operator $\hat{G}$ which is the inverse of the linear operator

$$(\hat{L}(u) + \lambda_0[f_y(y_0) + \delta_0 g_y(y_0)]u, \hat{B}(u)) \tag{19}$$

which occurs in any order ($\neq 0$) of the perturbation system (16). If $G(\mathbf{x}, \mathbf{x}\dagger)$ is the kernel of $\hat{G}$, then $G(\mathbf{x}, \mathbf{x}\dagger) < 0$ for any pair of values $\mathbf{x}$, $\mathbf{x}\dagger \in \Omega$ (comp. [7]). It is therefore obtained for the first order (16a):

$$y'(\mathbf{x}) = -\hat{G}\{\lambda_0 g(y_0) + \lambda'(f(y_0) + \delta_0 g(y_0))\}(\mathbf{x}). \tag{20}$$

λ' is then determined by the constraint (17) as:

$$\lambda' = -\lambda_0 \frac{\hat{G}(g(y_0))(\mathbf{x}_0)}{\hat{G}(f(y_0) + \delta_0 g(y_0))(\mathbf{x}_0)}. \tag{21}$$

Because of the negativity of the kernel of $\hat{G}$ and because of (13), one finds:

$$\hat{G}(f(y_0) + \delta_0 g(y_0))(\mathbf{x}) < 0 \quad \text{on } \Omega. \tag{22}$$

The higher order contributions for the series expansion of y and λ with respect to $(\delta - \delta_0)$ can be found in a similar fashion. If $(y_0, \lambda_0, \delta_0)$ represents a critical solution of (11), then the principal eigenvalue of the operator pair (19) vanishes and the related eigenfunction u_0 does not change sign on Ω. The value of λ' is fixed such that the first order problem (16b) can be solved:

$$\lambda' = -\frac{\lambda_0 \int_\Omega \mathrm{d}x\, u_0 g(y_0)}{\int_\Omega \mathrm{d}x\, u_0(f(y_0) + \delta_0 g(y_0))}. \tag{23}$$

The integral in the denominator of (23) is nonvanishing on account of $u_0 \neq 0$ on Ω and on account of (13). y' obtains from equation (16b) as:

$$y' = -\hat{G}(\lambda_0 g(y_0)$$
$$+ \lambda'(f(y_0) + \delta_0 g(y_0))) + c u_0. \tag{24}$$

The kernel of $\hat{G}$ is a generalization of Green's function. The constant c in the expression (24) is determined by the constraint (17):

$$c = \frac{1}{u_0(\mathbf{x}_0)} \hat{G}\{\lambda_0 g(y_0) + \lambda'(f(y_0) + \delta_0 g(y_0))(x_0) \tag{25}$$

The higher order contributions for the series expansion of y and λ with respect to $(\delta - \delta_0)$ can be found in a similar fashion.

The analytic continuation by regular perturbation with respect to the maximum y_m of the temperature y on $\Omega + \Gamma$ makes it possible to investigate for a given value of y_m appropriately defined functions which are minorants or majorants of $W(y)$ on the interval $0 \leqq y \leqq y_m$ only. Returning to the original problem (1), let $W(y)$ be a function which is known for values of y with $0 \leqq y \leqq \bar{y}$. For a given maximum value y_m with $0 < y_m \leqq \bar{y}$, the functions $m(y, y_m)$ and $M(y, y_m)$ are called positive minorant and positive majorant of $W(y)$ on the interval $0 \leqq y \leqq y_m$, if the following inequality holds:

$$0 < m(y, y_m) \leqq W(y) \leqq M(y, y_m)$$
$$\text{for} \quad 0 \leqq y \leqq y_m. \tag{26}$$

$m(y, y_m)$ and $M(y, y_m)$ are chosen such that the following boundary-value problems [comp. (1)] have unique positive solutions:

$$\hat{L}(v) + \lambda_m m(v, y_m) = 0 \quad \text{on } \Omega,$$

$$\hat{B}(v) = 0 \quad \text{on } \Gamma, \tag{27}$$

$$\hat{L}(u) + \lambda_M M(u, y_m) = 0 \quad \text{on } \Omega,$$
$$\hat{B}(u) = 0 \quad \text{on } \Gamma. \tag{28}$$

Here, $\lambda_m(y_m)$ and $\lambda_M(y_m)$ are determined by the following conditions:

$$\max_{\mathbf{x}\in\Omega+\Gamma} v(\mathbf{x}) = v(\mathbf{x}_0) = y_m, \tag{29}$$

$$\max_{\mathbf{x}\in\Omega+\Gamma} u(\mathbf{x}) = u(\mathbf{x}_0) = y_m. \tag{30}$$

By varying the parameter y_m, varied minorants $m(y, y_m)$ and majorants $M(y, y_m)$ result and one thus obtains functions $\lambda_m(y_m)$ and $\lambda_M(y_m)$. The following section is devoted to investigating the relation of the functions $\lambda_m(y_m)$ and $\lambda_M(y_m)$ to the function $\lambda(y_m)$ which is the response function of the thermodynamic system and part of the parametric representation (4) of the solution (y, λ) of problem (1).

For a given y_m, let it be defined on $0 \leqq y \leqq y_m$ that:

$$f(y) = m(y, y_m),$$
$$g(y) = W(y) - m(y, y_m) \geqq 0. \tag{31}$$

Then one obtains for any δ with $0 \leqq \delta \leqq 1$:

$$f(y) + \delta g(y)$$
$$= (1-\delta)m(y, y_m) + \delta W(y) > 0. \tag{32}$$

Thus, condition (13) is satisfied. For a δ_0 with $0 \leqq \delta_0 \leqq 1$, let $(y_0, \lambda_0, \delta_0)$ be a stable solution of (11). Then one obtains on account of the negativity of the kernel of $\hat{G}$ and on account of equations (31) and (32) from (21):

$$\lambda' \leqq 0. \tag{33}$$

If $(y_0, \lambda_0, \delta_0)$ designates a critical solution of (11) for a δ_0 with $0 \leqq \delta_0 \leqq 1$, then one derives from (23) on account of (31) and (32) and on account of $u_0 \neq 0$ on Ω:

$$\lambda' \leqq 0. \tag{34}$$

For $\delta_0 = 0$, the solution $(v, \lambda_m(y_m))$ of (27), (29) is obtained, for $\delta_0 = 1$, the solution $(y, \lambda(y_m))$ of (1) results. If analytic continuation from $(v, \lambda_m(y_m))$ to $(y, \lambda(y_m))$ is possible from $\delta = 0$ to $\delta = 1$ and if this path leads through stable solutions of (11) only or if it stays close to critical solutions of (11), then one obtains from (33), (34):

$$\lambda_m(y_m) \geqq \lambda(y_m). \tag{35}$$

For the given value of y_m, let it now be defined on $0 \leqq y \leqq y_m$ that:

$$f(y) = M(y, y_m),$$
$$g(y) = W(y) - M(y, y_m) \leqq 0. \tag{36}$$

Then the following property is found to hold for $0 \leqq \delta \leqq 1$:

$$f(y) + \delta g(y) = (1-\delta)M(y, y_m) + \delta W(y) > 0. \tag{37}$$

By virtue of (37), condition (13) is fulfilled. For $0 \leqq \delta_0 \leqq 1$, let $(y_0, \lambda_0, \delta_0)$ be a stable solution of (11). Then one obtains on account of the negativity of the kernel of $\hat{G}$ and on account of (36), (37) from equation (21):

$$\lambda' \geqq 0. \tag{38}$$

If $(y_0, \lambda_0, \delta_0)$ designates a critical solution of (11) for $0 \leqq \delta_0 \leqq 1$, then one derives from equation (23) on account of (36), (37) and on account of $u_0 \neq 0$ on Ω:

$$\lambda' \geqq 0. \tag{39}$$

For $\delta_0 = 0$, the solution $(u, \lambda_M(y_m))$ of (28), (30) is obtained, for $\delta_0 = 1$, the solution $(y, \lambda(y_m))$ of (1) results. If analytic continuation is possible from $\delta = 0$ to $\delta = 1$ and if this path leads through stable solutions of (11) only, or stays sufficiently close to critical solutions of (11), then one obtains from (38), (39):

$$\lambda_M(y_m) \leqq \lambda(y_m). \tag{40}$$

By varying y_m continuously from $y_m = 0$ till $y_m = \bar{y}$ and by varying the minorants $m(y, y_m)$ and the majorants $M(y, y_m)$ in a continuous fashion with respect to their dependence on y_m accordingly, one obtains under the above assumptions continuous functions $\lambda_m(y_m)$ and $\lambda_M(y_m)$ which provide upper and lower bounds of $\lambda(y_m)$ for any value y_m with $0 \leqq y_m \leqq \bar{y}$:

$$\lambda_M(y_m) \leqq \lambda(y_m) \leqq \lambda_m(y_m). \tag{41}$$

4. LINEAR POSITIVE MINORANTS AND MAJORANTS

Functions $m(y, y_m)$ and $M(y, y_m)$ which are linear in y on $0 \leqq y \leqq y_m$ and which satisfy equation (26) are of particular importance because for linear functions, the problems (27) and (28) can be solved by standard methods, irrespective of whether $\hat{L}$ is in the form of an ordinary or a partial differential operator. In order to define a pointwise scanning of the given function $W(y)$ of problem (1) by positive linear minorants and majorants, the following additional property is imposed:

$$m(y_m, y_m) = W(y_m) = M(y_m, y_m). \tag{42}$$

It is by virtue of equation (42) that for a given value of y_m, $0 < y_m \leqq \bar{y}$, a linear minorant and a linear majorant of $W(y)$ on $0 \leqq y \leqq y_m$ can be found which— while satisfying equation (26)— are unique in being the closest to $W(y)$. The geometrical construction of the closest linear minorant and the closest linear majorant for the given value of y_m is straightforward if the graph of $W(y)$ is given, so that for the construction of the functions $\lambda_m(y_m)$ and $\lambda_M(y)$ the actual analytical expression of $W(y)$ does not need to be at hand (comp. [8] for examples).

In order to provide an example of how $\lambda_M(y_m)$ can be constructed, let it be assumed that $W(y)$ is a monotonely increasing function in y with $W(y) > 0$ for $0 \leqq y \leqq \bar{y}$. Let y_m be given with $0 < y_m \leqq \bar{y}$. Then it is possible to find a constant positive majorant $M(y, y_m) = W(y_m)$, which satisfies (26) and (42). Subtracting eqution (28) from (1), one obtains, on account of the linearity of $\hat{L}$ and $\hat{B}$:

$$\hat{L}(u-y) + \lambda_M W(y_m) - \lambda W(y) = 0 \quad \text{on } \Omega,$$
$$\hat{B}(u-y) = 0 \quad \text{on } \Gamma. \quad (43)$$

Let the symmetry defined position $\mathbf{x}_0$ of the maximum y_m of y and u be an interior point. Imposing (5) and (30) in order to obtain $\lambda(y_m)$ and $\lambda_M(y_m)$, one finds at the interior point $\mathbf{x}_0 \in \Omega$:

$$u(\mathbf{x}_0) - y(\mathbf{x}_0) = 0, \quad (44)$$

$W(y)$ being increasing in y, one derives that:

$$W(y) \leqq W(y_m) \quad \text{for} \quad 0 \leqq y \leqq y_m. \quad (45)$$

On account of the maximum principle for elliptic operators [7], it is then deduced that one must necessarily have:

$$\lambda_M(y_m) \leqq \lambda(y_m). \quad (46)$$

By varying y_m between $y_m = 0$ and $y_m = \bar{y}$, one is thus able to derive a continuous function $\lambda_M(y_m)$ which provides a lower bound of $\lambda(y_m)$ for any y_m with $0 \leqq y_m \leqq \bar{y}$.

For a function $W(y)$ with $W(y) > 0$ in $0 \leqq y \leqq \bar{y}$, it is possible to find a linear 'closest' majorant $M(y, y_m)$ for any value of y_m with $0 < y_m \leqq \bar{y}$, such that $M(y, y_m)$ satisfies equations (26) and (42). For a function $W(y)$ with $W(y) > 0$ in $0 \leqq y \leqq \bar{y}$, there always exists a subinterval such that a positive linear minorant $m(y, y_m)$ exists for any y_m from this subinterval, while $m(y, y_m)$ satisfies (26) and (42). The tangent $t(y, y_m)$ of $W(y)$ for $y = y_m$ results as:

$$t(y, y_m) = (y - y_m)W_y(y_m) + W(y_m). \quad (47)$$

For a function $W(y)$ with $W(y) > 0$, there always exists an interval $0 \leqq y_m \leqq y^*$, $y^* \leqq \bar{y}$, such that the tangent $t(y, y_m)$ [comp. (47)] for any y_m with $0 \leqq y_m \leqq y^*$ intersects the W-axis for a positive W-value:

$$W(y_m) - y_m W_y(y_m) > 0 \quad \text{for} \quad 0 \leqq y_m \leqq y^*. \quad (48)$$

For y_m with $0 < y_m \leqq y^*$, then there exists—by inspection—a linear closest positive minorant satisfying (26) and (42). On account of the property (48), the function $W(y)$ may be called 'concave in a generalized sense' [comp. [14]] for $0 \leqq y \leqq y^*$ and it may be shown that for a maximum y_m with $0 < y_m \leqq y^*$, the solution of equation (1) exists and is stable.

Thus, for y_m with $0 \leqq y_m \leqq y^*$, linear positive majorants $M(y, y_m)$ and linear positive minorants $m(y, y_m)$ of $W(y)$ exists and it is easily seen that the function $(1-\delta)m(y, y_m) + W(y)$ [comp. (32)] and the function $(1-\delta)M(y, y_m) + W(y)$ [comp. (37)] are concave in the sense of (48) for $0 \leqq \delta \leqq 1$. It is therefore concluded that the expression (41) holds for any y_m with $0 \leqq y_m \leqq y^*$, $y^* \leqq \bar{y}$, i.e. that the expression

$$\lambda_M(y_m) \leqq \lambda(y_m) \leqq \lambda_m(y_m)$$

holds on the interval $0 \leqq y_m \leqq y^*$, where the function $W(y)$ is concave in the sense of equation (48).

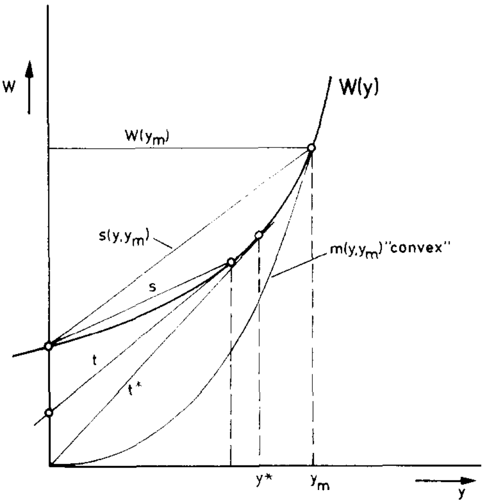

Fig. 1. Positive minorants and majorants of the function $W(y) = \exp(y)$.

5. A NUMERICAL EXAMPLE

In order to give a demonstration of how the proposed method of approximation applies, the following problem is investigated:

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \lambda \exp(y) = 0,$$
$$y(1) = 0, \quad y(0) = y_m \quad (49)$$
$$\left.\frac{dy}{dx}\right|_{x=0} = 0.$$

For this problem, a solution exists in the parametric form (4) which has been derived numerically by [12, 15]. By using their result, it is possible to test the continuous upper and lower bounds $\lambda_m(y_m)$ and $\lambda_M(y_m)$ which derive from the introduction of appropriate minorants and majorants. The function $W(y) = \exp(y)$ is concave in the sense of (48) for y with $0 \leqq y < y^* = 1$, so that for a given y_m with $0 < y_m < 1$, uniquely defined linear minorants and majorants exist which satisfy (26) and (42) and which are 'closest' to $W(y)$. Specifically, the closest linear majorant is the secant $s = s(y, y_m)$ passing through $W = 1$ and $W(y_m) = \exp(y_m)$, whereas the closest linear minorant is the tangent $t = t(y, y_m)$ for $W(y)$ for $y = y_m$ (comp. Fig. 1).

The problems (27) and (29) and (28) and (30) may be solved with $m(y, y_m) = t(y, y_m)$, $M(y, y_m) = s(y, y_m)$ and the resulting bounds are designated $\lambda_m(y_m)$ 'tangent' and $\lambda_M(y_m)$ 'secant' and may be compared with the exact solution designated $\lambda(y_m)$ 'exact' (comp. Fig. 2). For $y_m \geqq y^*$, the curve $\lambda_m(y_m)$ 'tangent' is parallel to the y_m axis and defined by the value λ^* derived from the tangent t^* in Fig. 1 (comp. [16]).

$W(y) = \exp(y)$ is monotonely increasing in y. Thus, constant linear majorants $M(y, y_m) = \exp(y_m)$ may be used in problem (28), (30) to give a curve of bounds designated $\lambda_M(y_m)$ 'constant' in Fig. 2. By using the results of [17] for Emden's equation, convex nonlinear minorants $m(y, y_m) = (y^2/y_m^2)\exp(y_m)$ (marked $m(y, y_m)$

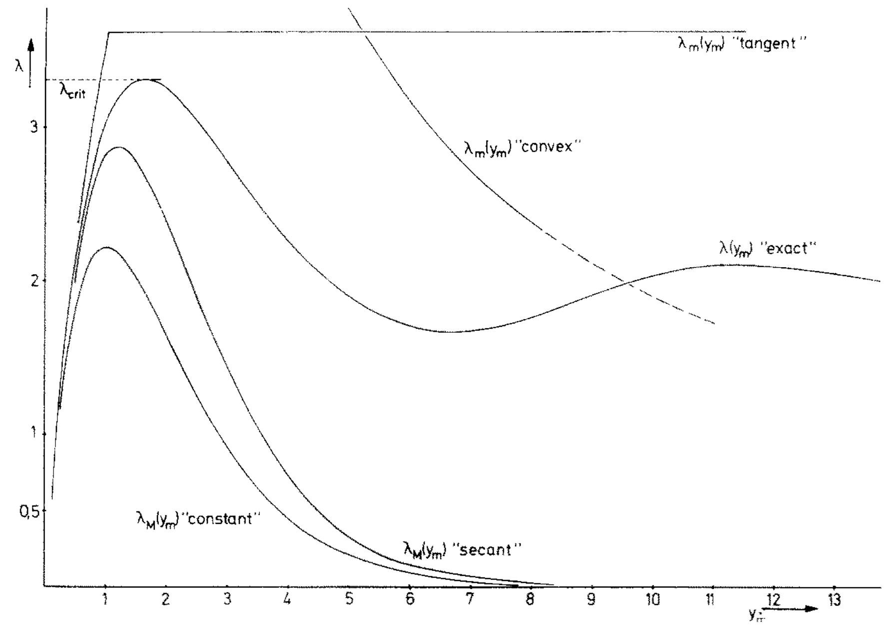

FIG. 2. The exact solution $\lambda(y_m)$ and the continuous upper and lower bounds $\lambda_m(y_m)$ and $\lambda_M(y_m)$.

'convex' in Fig. 1) may be introduced for the derivation—by similarity methods—of improved upper bounds which are marked $\lambda_m(y_m)$ 'convex' in Fig. 2.

Finally, it should be emphasized that stable solutions of problem (49) occur only for values of y_m between $y_m = 0$ and $y_m = 160\,746$, which is related to the first maximum of the response curve $\lambda(y_m)$ (comp. [12]). If one recalls that only stable or critical solutions of problem (1) are of physical interest, Fig. 2 shows that the proposed method of approximation provides good bounds for the physically interesting part of $\lambda(y_m)$. In particular, good bounds are obtained by a relatively simple method for the critical values of the parameter λ (marked λ_{crit} in Fig. 2), which are of considerable interest (e.g. Frank-Kamenetzki's theory of thermal explosions, comp. [9, 18]).

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UNE METHODE D'APPROXIMATION MINORANTE-MAJORANTE POUR LA SOLUTION DE L'EQUATION DE FOURIER NON LINEAIRE

Résumé—Dans une étude de l'équation non linéaire de Fourier par une méthode analytique basée sur la température maximale y_m comme paramètre de continuation, on montre comment des majorants positifs et des minorants positifs (qui varient avec y_m) peuvent être utilisés pour fournir des limites continues supérieures et inférieures d'une fonction $\lambda(y_m)$ qui gouverne l'intensité de la génération de chaleur au cours d'un certain processus exothermique. La fonction $\lambda(y_m)$ constitue une propriété caractéristique du problème car elle contient l'information d'existence de distribution de température stable, instable ou critique qui résulte du problème des conditions aux limites de l'équation de Fourier. De façon à fournir un test de la méthode proposée, des limites continues supérieure et inférieure de $\lambda(y_m)$ sont obtenues pour un problème dont on connait la solution numérique.

Zusammenfassung—Untersucht man die nichtlineare Fouriersche Gleichung der Wärmeleitung mit Hilfe der Methode der analytischen Fortsetzung, so läßt sich die maximale Temperatur y_m als Fortsetzungsparameter wählen. Dann lassen sich von y_m abhängende positive Majoranten und positive Minoranten einführen, durch welche in y_m stetige obere und untere Schranken einer Funktion $\lambda(y_m)$ bestimmt werden, die die Intensität der Wärmeerzeugung für die untersuchten exothermen Prozesse steuert. Die Funktion $\lambda(y_m)$ stellt eine charakteristische Größe des Problems dar, da sie Aussagen enhält über die Existenz stabiler, instabiler und kritischer Temperaturverteilungen, welche man als Lösungen des Randwertproblems der Fourierschen Gleichung erhält. Um die Güte der vorgestellten Methode zu demonstrieren, werden in y_m stetige obere und untere Schranken von $\lambda(y_m)$ für ein Fouriersches Problem berechnet, dessen exakte numerische Lösung bekannt ist.

ПРИМЕНЕНИЕ МЕТОДА МИНОРАНТНЫХ-МАЖОРАНТНЫХ АППРОКСИМАЦИЙ ПРИ РЕШЕНИИ НЕЛИНЕЙНОГО УРАВНЕНИЯ ФУРЬЕ

Аннотация — При анализе нелинейного уравнения теплопроводности Фурье методом аналитического продолжения с максимумом температуры y_m в качестве параметра продолжения показано, каким образом положительные мажоранты и миноранты, изменяющиеся с изменением параметра y_m, можно использовать для определения непрерывных верхних и нижних границ функции $\lambda(y_m)$, описывающей интенсивность тепловыделения в результате некоторого экзотермического процесса. Функция $\lambda(y_m)$ содержит информацию о наличии устойчивого, неустойчивого и критического распределений температуры, полученных при решении граничной задачи для уравнения Фурье. Для проверки предложенного метода определены непрерывные верхние и нижние границы функции $\lambda(y_m)$ задачи Фурье, для которой имеется точное численное решение.